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An analytical solution to the nonlinear model for N trapped ions driven by a single laser field

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Abstract
We have studied the nonlinear model for interaction of N ions in a trap with a laser beam. By applying two successive unitary transformations, this nonlinear quantum system can be transformed into the multi-levels Jayness–Cummings model. In the end, we can obtain an analytical solution to the two-ion system without performing the Lamb–Dicke approximation, and observe the super-revival phenomenon.

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1. Introduction

The investigation of trapped ions driven by light fields is of importance not only due to the fundamental quantum optics, but also because of potential application, such as precision spectroscopy [1] and quantum computation [2]. The dynamics of a trapped ion with a single mode of the radiation field has been shown to exhibit interesting new phenomena, such as the collapse and revivals of Rabi oscillations [3], squeezing of vibrational states of ion [4], generation of Schrödinger cat states of ion [5], entanglement of trapped ions [6, 7], etc. These features, analogous with those in the Jayness–Cummings model (JCM) of cavity quantum electric dynamics, are directly related to the discreteness of the vibrational states in an ion trap. In processing quantum systems of trapped ions with the radiation fields, the Lamb–Dicke approximation is commonly used, i.e. the vibrational amplitude of the ion centre-of-mass motion is required to be much less than the associated optical wavelength.

Recently an alternative approach is proposed to avoid constraint on the Lamb–Dicke parameter in studying the system of trapped ion-laser [8]. In this approach, the Hamiltonian of the system can be transformed to usual the Jayness–Cummings model in context of cavity QED by using a unitary transformation.

In this paper, we investigate quantum system of N two-level trapped ions driven by a laser beam. It is well known that this quantum system is nonlinear beyond the Lamb–Dicke regime.

We find that this nonlinear quantum system is transformed into N levels JCM by applying unitary transformations.

2. Linearization of the Hamiltonian

We consider N two-level ions trapped in a harmonic potential trap, which are illuminated homogeneously by a laser field with frequency ω_L and wavenumber k_L. We restrict our consideration to the quantum-mechanical motion of the ion in the x-direction. We consider the Coulomb interaction between N ions and the variation of the field mode within the ion system, i.e. allow for the centre-of-mass wavefunction being large compared to the wavelength. With application of the optical rotating wave approximation, the Hamiltonian is given as

$$H = H_t + H_a + H_{\text{int}} \tag{1}$$

where

$$H_t = \frac{1}{2m}\sum_{j=1}^{N} p_j^2 + \frac{m}{2}\sum_{j=1}^{N}\omega^2 x_j^2 + \frac{e^2}{4\pi\varepsilon_0}\sum_{k<j}^{N}\frac{1}{|x_j - x_k|} \tag{2}$$

$$H_a = \frac{\hbar\Delta}{2}\sum_{j=1}^{N}|e_j\rangle\langle e_j| \tag{3}$$

$$H_{\text{int}} = \frac{\hbar\Omega_0}{2}\sum_{j=1}^{N}\mathrm{e}^{\mathrm{i}k_L x_j}\,|e_j\rangle\langle g_j| + \text{h.c.} \tag{4}$$

and $\Delta = \omega_a - \omega_L$ is the detuning of the ionic transition from the laser frequency. In the harmonic approximation, we express H_t as a function of the displacements around the equilibrium positions ($x_j = x_{j0} + r_j$), up to quadratic terms,

$$H_t = \frac{1}{2m}\sum_{j=1}^{N} p_j^2 + \frac{m\omega^2}{2}\sum_{i,j=1}^{N} K_{ij} r_i r_j \tag{5}$$

where

$$K_{ij} = \begin{cases} 1 + 2\displaystyle\sum_{j'(\neq i)}\frac{l^3}{|x_{j0} - x_{j'0}|^3}, & i = j \\ -2\dfrac{l^3}{|x_{i0} - x_{j0}|^3}, & i \neq j \end{cases} \tag{6}$$

and $l = [e^2/(4\pi\varepsilon_0 m\omega^2)]^{1/3}$ is the scale length. The corresponding Hamiltonian can be diagonalized in terms of normal modes. Applying the normal transformation, $r_j = \sum_{m=1}^{N} M_{jm} Q_m$, $p_j = \sum_{m=1}^{N} M_{jm} P_m$, Q_m, P_m is the coordinates and momentum of the normal mode, respectively, and M_{jm} is the relation (a matrix) between the relative position of the ith ion and the normal mode coordinates, $\sum_{i,j} M_{in} K_{ij} M_{jm} = \omega_n^2 \delta_{nm}$, $\sum_j M_{jm} M_{jn} = \delta_{nm}$, we obtain the transformed Hamiltonian H_t as

$$H_t = \sum_{j=1}^{N}\frac{P_j^2}{2m} + \frac{m}{2}\sum_{j=1}^{N}\omega_j Q_j^2. \tag{7}$$

Local coordinates can be expressed in terms of creation and annihilation of normal modes:

$$Q_i = \sum_n \frac{M_{in}}{\sqrt{2m\omega_n/\hbar}}\left(a_n^+ + a_n\right) \tag{8}$$

where $\hat{a}_n = (m\omega_n/2\hbar)^{1/2}(Q_n + \mathrm{i}P_n/m\omega_n)$ is the usual annihilation operator of the nth normal mode of the ion. Under above consideration, the interaction Hamiltonian can be expressed as

$$H_{\rm int} = \frac{\hbar\Omega}{2}\sum_{j=1}^{N}\exp\left(\mathrm{i}\sum_{m=1}^{N}\eta_m M_{jm}\left(a_m^+ + a_m\right)\right)|e_j\rangle\langle g_j| + \text{h.c.} \tag{9}$$

where $\Omega = \Omega_0\,\mathrm{e}^{\mathrm{i}k_L x_{j0}}$. We start by taking the frequency of laser resonant with the centre-of-mass vibronic transition in the k-th blue or red sideband as $\Delta = \omega_a - \omega_L = \pm k\nu$, ν is the frequency of the centre-of-mass vibronic motion. For small k values, we may safely assume that only the centre-of-mass motion will be excited, given that the next eigenfrequency is $\nu_r = \sqrt{3}\nu$, corresponding to the breathe mode of the ions [9]. The following frequencies ($\geqslant\sqrt{29/5}\nu$) depend on the number of ions. In the following, we only consider the centre-of-mass vibronic motion. Assuming that $m = 1$ denotes the mode of the centre-of-mass motion, and noticing that the elements of the matrix M_{j1} are the same for all j, i.e., the centre-of-mass mode characterized by all ions has the same excursions [10]; we will obtain the Hamiltonian of the ions–laser system as [11]

$$\hat{H} = \hat{H}_t + \hbar\frac{\Delta}{2}\hat{J}_z + \frac{\hbar\Omega}{2}\left(\mathrm{e}^{-\mathrm{i}\eta(\hat{a}+\hat{a}^+)}\,\hat{J}_+ + \mathrm{e}^{\mathrm{i}\eta(\hat{a}+\hat{a}^+)}\,\hat{J}_-\right) \tag{10}$$

where

$$\hat{H}_t = \hbar\nu\hat{a}^+\hat{a} \tag{11}$$

$$\hat{J}_z = \sum_{i=1}^{N}\sigma_z^{(i)} \tag{12}$$

$$\hat{J}_+ = \sum_{i=1}^{N}\sigma_+^{(i)} \tag{13}$$

$$\hat{J}_- = \sum_{i=1}^{N}\sigma_-^{(i)} \tag{14}$$

$\eta = k\sqrt{\hbar/2m\nu}$ is the Lamb–Dick parameter, $\sigma_z^{(i)} = |e_i\rangle\langle e_i|$, $\sigma_+^{(i)} = |e_i\rangle\langle g_i|$ and $\sigma_-^{(i)} = |g_i\rangle\langle e_i|$ are Pauli's spin operators of the i-th ion. The 'angular momentum' operators $\hat{J}_z$, $\hat{J}_+$ and $\hat{J}_-$ form $SU(2)$ algebras with commutation relations:

$$[\hat{J}_z, J_\pm] = \pm 2\hat{J}_\pm \tag{15}$$

$$[\hat{J}_+, J_-] = \hat{J}_z. \tag{16}$$

We denote the 'angular momentum' states as $|J, M\rangle$, which in the present case are just the well-known Dicke states [12], and $J = N/2$ is the cooperation number. $|J, -J\rangle$ indicates a state for which all ions are in the ground state while for $|J, J\rangle$ all ions are in the excited states. For convenience, we number the state vectors $|J, M\rangle$ into $|i\rangle = |J, J - i + 1\rangle$, $(1 \leqslant i \leqslant K, K = 2J + 1)$, which satisfies

$$\hat{J}_+|i\rangle = \hbar\sqrt{(2J - i + 2)(i - 1)}|i - 1\rangle \tag{17}$$

$$\hat{J}_-|i\rangle = \hbar\sqrt{(2J - i + 1)i}|i + 1\rangle \tag{18}$$

It should be noted that we may investigate the dynamics of the system in the subspace spanned by Dicke states as long as the initial states of the system are located in this subspace.

In the representation of Dicke state vectors $|i\rangle$, the matrix elements of the Hamiltonian (1) can be rewritten as ($\hbar = 1$)

$$\langle i|\hat{H}|j\rangle = \hat{H}_t\delta_{i,j} + \frac{\Delta_j}{2}\delta_{i,j} + \frac{\Omega}{2}(\hat{D}^{+}(\alpha)A_{j+1}\delta_{i,(j+1)} + \hat{D}(\alpha)A_j\delta_{i,(j-1)}) \tag{19}$$

where $\hat{D}(\alpha) = \mathrm{e}^{(\alpha\hat{a}-\alpha\hat{a}^{+})}$ is the displacement operator, $\alpha = -\mathrm{i}\eta$, $A_i = \sqrt{(2J-i+2)(i-1)}$, $\Delta_j = (J-j+1)\Delta$. We note that the system described by the Hamiltonian (10) is nonlinear and its exact general solutions are very difficult to obtain, but we find that this system can be solved analytically. In order to obtain its analytical solution, we firstly operate a unitary transformation $\hat{T}_1$

$$\hat{T}_1 = \begin{pmatrix} \hat{D}(\beta_1) & 0 & 0 & \cdots & 0 \\ 0 & \hat{D}(\beta_2) & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \hat{D}(\beta_{K-1}) & 0 \\ 0 & \cdots & 0 & 0 & \hat{D}(\beta_K) \end{pmatrix} \tag{20}$$

to the Hamiltonian (10), where $\beta_i = \lambda_i\alpha/2$, $\lambda_i = 2\mathrm{i} - K - 1$, then obtain the following linearized Hamiltonian $\hat{H}' = T_1\hat{H}T_1^{+}$,

$$\hat{H}' = \begin{pmatrix} \hat{H}_t^{(1)} & 0 & 0 & \cdots & 0 \\ 0 & \hat{H}_t^{(2)} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \hat{H}_t^{(K-1)} & 0 \\ 0 & \cdots & 0 & 0 & \hat{H}_t^{(K)} \end{pmatrix} + \frac{\Delta}{2}\begin{pmatrix} J & 0 & 0 & \cdots & 0 \\ 0 & J-1 & 0 & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & J-K+1 & 0 \\ 0 & \cdots & 0 & 0 & -J \end{pmatrix}$$

$$\frac{\Omega}{2}\begin{pmatrix} 0 & A_2 & 0 & \cdots & 0 \\ A_2 & 0 & A_3 & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & A_{K-1} & 0 & A_K \\ 0 & \cdots & 0 & A_K & 0 \end{pmatrix} \tag{21}$$

where $\hat{H}_t^{(i)} = \hat{D}(\beta_i)\hat{H}_t\hat{D}^{+}(\beta_i)$. In order to diagonalize the third matrix in (21), we furthermore apply a unitary transformation to the Hamiltonian (21), which has the form of

$$\hat{T}_2 = \begin{pmatrix} x_1^{(1)} & x_2^{(1)} & \cdots & x_{K-1}^{(1)} & x_K^{(1)} \\ x_1^{(2)} & x_2^{(2)} & \cdots & x_{K-1}^{(2)} & x_K^{(2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_1^{(K-1)} & x_2^{(K-1)} & \cdots & x_{K-1}^{(K-1)} & x_K^{(K-1)} \\ x_1^{(K)} & x_2^{(K)} & \cdots & x_{K-1}^{(K)} & x_K^{(K)} \end{pmatrix} \tag{22}$$

where $x_j^{(i)}$ satisfies the eigen equations

$$A_i x_{i-1}^{(\mu)} + A_{i+1}x_{i+1}^{(\mu)} = \lambda^{(\mu)}x_i^{(\mu)}. \tag{23}$$

It can be proven that the eigenvalues [12]

$$\lambda^{(\mu)} = N - 2\mu + 2 \tag{24}$$

while the eigenvectors $x_j^{(i)}$ can be expressed in the compact form [12]

$$x_j^{(i)} = 2^{-J+j}\left[\frac{j!(2J-j)!}{i!(2J-i)!}\right]^{1/2} \times P_j^{(i-j),(2J-i-j)}(0) \tag{25}$$

while $P_l^{(m,n)}$ is the Jacobian polynomials,

$$P_l^{(m,n)}(z) = \left[\frac{(-1)^l}{2^l l!}\right][(1-z)^m(1+z)^n]^{-1} \times \frac{\mathrm{d}^l}{\mathrm{d}z^l}[(1-z)^{m+l}(1+z)^{n+l}]. \quad (26)$$

By taking advantage of the orthogonality of the eigenvectors $x_j^{(i)}$, the matrix elements of the transformed Hamiltonian $\hat{H}'' = \hat{T}_2\hat{H}'\hat{T}_2^+$ can be easily computed, which is represented as

$$\langle i|\hat{H}''|i\rangle = \nu\hat{a}^+\hat{a} + \nu|\alpha|^2\left(A_i^2 + A_{i+1}^2\right)/4 + \Omega\lambda_i/2 \quad (27a)$$

$$\langle i|\hat{H}''|j\rangle = g_{i,j}(\hat{a} - \hat{a}^+) + F_{i,j} \qquad (j \neq i) \quad (27b)$$

where

$$g_{i,j} = -\frac{\alpha\nu}{2}(A_j\delta_{i,(j-1)} + A_{j+1}\delta_{i,(j+1)}) \quad (28)$$

$$F_{i,j} = \frac{\Delta}{4}(A_i\delta_{(i-1),j} + A_{i+1}\delta_{(i+1),j}) + \frac{|\alpha|^2\nu}{4}(A_iA_{j+1}\delta_{(i-1),(j+1)} + A_{i+1}A_j\delta_{(i+1),(j-1)}). \quad (29)$$

It is noted that this Hamiltonian is linear and holds for any value of the Lamb–Dicke parameter η. We can see that the nonzero elements of the matrices $g_{i,j}$ and $F_{i,j}$ are, respectively,

$$g_{i,i+1} = g_{i+1,j} = -\frac{\alpha\nu}{2}A_{i+1} \quad (30)$$

$$F_{i,i+2} = F_{i+2,i} = \frac{|\alpha|^2\nu}{4}A_{i+1}A_{i+2} \quad (31a)$$

$$F_{i,i+1} = F_{i+1,i} = \frac{\Delta}{4}A_{i+1}. \quad (31b)$$

3. Solution to the model

In order to obtain a solution to the model described by the Hamiltonian (10), we first derive the time evolution operator corresponding to the Hamiltonian (27). For this goal, noting that $\lambda_i = -\lambda_{K+2-i}$, we may rewrite $\hat{H}''$ into

$$\hat{H}'' = H_0'' + H_I'' \quad (32)$$

where

$$\hat{H}_0'' = \nu\hat{a}^+\hat{a} + \frac{\Omega}{2}\sum_{i=1}^{K'}\lambda_i(|K+2-i\rangle\langle K+2-i| - |i\rangle\langle i|) \quad (33)$$

$$\hat{H}_I'' = \sum_{i=1}^{K}\delta_i|i\rangle\langle i| + \left\{\sum_{i=1}^{K-1} g_{i,(i+1)}(\hat{a} - \hat{a}^+)|i\rangle\langle i+1| + \mathrm{h.c.}\right\}$$
$$+ \left\{\sum_{i=1}^{K-1} F_{i,i+1}|i\rangle\langle i+1| + \sum_{i=1}^{K-2} F_{i,i+2}|i\rangle\langle i+2| + \mathrm{h.c.}\right\} \quad (34)$$

$$\delta_i = \nu|\alpha|^2\left(A_i^2 + A_{i+1}^2\right)/4. \quad (35)$$

It is noted that $K' = K/2$ for which K is even, $K' = (K+1)/2$ for K odd. After we perform a unitary transformation $\tilde{\hat{H}}_I = \hat{U}_0 \hat{H}_I'' \hat{U}_0^+$, $U_0 = \exp\{-\mathrm{i}t H_0''\}$, the interaction Hamiltonian $\hat{H}_I''$ is transformed as

$$\hat{H}_I'' = \sum_{i=1}^{K} \delta_i |i\rangle\langle i| + \left\{ \sum_{i=1}^{K-1} g_{i,(i+1)} (\hat{a}\,\mathrm{e}^{-\mathrm{i}\nu t} - \hat{a}^+ \mathrm{e}^{\mathrm{i}\nu t})\,\mathrm{e}^{\mathrm{i}\Omega t}\, |i\rangle\langle i+1| + \text{h.c.} \right\}$$
$$+ \left\{ \sum_{i=1}^{K-2} F_{i,i+2}\,\mathrm{e}^{\mathrm{i}2\Omega t}\, |i\rangle\langle i+2| + \sum_{i=1}^{K-1} F_{i,(i+1)}\,\mathrm{e}^{\mathrm{i}\Omega t}\, |i\rangle\langle i+1| + \text{h.c.} \right\}. \tag{36}$$

In what follows, for simplicity we set $\Omega = \nu$. Neglecting the rapidly oscillating terms in (36), we then obtain

$$\hat{H}_I'' = \sum_{i=1}^{K} \delta_i |i\rangle\langle i| + \left\{ \sum_{i=1}^{K-1} G_i \hat{a} |i\rangle\langle i+1| + \text{h.c.} \right\} \tag{37}$$

where $G_i = g_{i,i+1} = g_{i+1,j} = -\frac{\alpha\nu}{2} A_{i+1}$. It is noted that the Hamiltonian (37) is a multi-level Jaynes–Commings Hamiltonian, which has been discussed extensively [13].

We now proceed to give a solution to the Hamiltonian (10), which reads

$$|\psi(t)\rangle = \hat{T}^+ \hat{U}_0 \hat{U}_I \hat{T} |\psi(0)\rangle \tag{38}$$

where $|\psi(0)\rangle$ is the initial wave vector, $\hat{T} = \hat{T}_2 \hat{T}_1$, $\hat{U}_I$ is the evolution operator generated by the Hamiltionian (37).

As an example, we consider the case of two ions, i.e. $N = 2$. According to equations (33) and (37), the transformed Hamiltonian of two ions with a laser beam is represented in the form of

$$\hat{H}'' = \nu \hat{a}^+ \hat{a} + \nu(|1\rangle\langle 1| - |3\rangle\langle 3|) + K|2\rangle\langle 2| + \{\mathrm{i}g\hat{a}(|1\rangle\langle 2| + |2\rangle\langle 3|) + \text{h.c.}\} \tag{39}$$

where $K = \nu|\alpha|^2/2$, $g = \sqrt{2}\nu\eta/2$, $|1\rangle = |e_1 e_2\rangle$, $|2\rangle = (|g_1 e_2\rangle + |e_1 g_2\rangle)/\sqrt{2}$, $|3\rangle = |g_1 g_2\rangle$. It is noted that we have disregarded the constant term in the Hamiltonian (39) because it just represents an overall phase. The time evolution operator $\hat{U}$ corresponding to the Hamiltonian (39) is easily obtained, which is given by

$$\hat{U} = \exp(-\mathrm{i}\nu t - \mathrm{i}\nu t(|1\rangle\langle 1| - |3\rangle\langle 3|)) \exp(-\mathrm{i}Kt|2\rangle\langle 2| + gt\{\hat{a}|2\rangle\langle 3| + |1\rangle\langle 2| - \text{h.c.}\})$$
$$= \exp(-\mathrm{i}\nu t\hat{n} - \mathrm{i}Kt/2) \left\{ \sum_{i,j=1}^{3} \hat{U}_{ij} |i\rangle\langle j| \right\} \tag{40}$$

where

$$\hat{U}_{11} = \mathrm{e}^{\mathrm{i}Kt/2}\, \hat{B}^+ \hat{B} + \hat{A} \left(\cos(\hat{f}_{\hat{n}} t) + \mathrm{i}K \frac{1}{\hat{f}_{\hat{n}}} \sin(\hat{f}_{\hat{n}} t) \right) \hat{A}^+ \tag{41}$$

$$\hat{U}_{12} = -\mathrm{i}\hat{A} \frac{1}{\hat{f}_{\hat{n}}} \sin(\hat{f}_{\hat{n}} t) \tag{42}$$

$$\hat{U}_{13} = \hat{A} \left[-\mathrm{e}^{\mathrm{i}Kt} + \cos(\hat{f}_{\hat{n}} t) + \mathrm{i}K \frac{1}{\hat{f}_{\hat{n}}} \sin(\hat{f}_{\hat{n}} t) \right] \hat{B} \tag{43}$$

$$\hat{U}_{21} = -\mathrm{i} \frac{1}{\hat{f}_{\hat{n}}} \sin(\hat{f}_{\hat{n}} t) \hat{A} \tag{44}$$

$$\hat{U}_{22} = \left\{ \cos(\hat{f}_{\hat{n}} t) - \mathrm{i}K \frac{1}{\hat{f}_{\hat{n}}} \sin(\hat{f}_{\hat{n}} t) \right\} \tag{45}$$

$$\hat{U}_{23} = -\mathrm{i}\frac{1}{\hat{f}_{\hat{n}}}\sin(\hat{f}_n t)\hat{B} \tag{46}$$

$$\hat{U}_{31} = \hat{B}^{+}\left[-\mathrm{e}^{\mathrm{i}Kt} + \cos(\hat{f}_{\hat{n}}t) + \mathrm{i}K\frac{1}{\hat{f}_{\hat{n}}}\sin(\hat{f}_{\hat{n}}t)\right]\hat{A}^{+} \tag{47}$$

$$\hat{U}_{32} = -\mathrm{i}\hat{B}^{+}\frac{1}{\hat{f}_{\hat{n}}}\sin(\hat{f}_{\hat{n}}t) \tag{48}$$

$$\hat{U}_{33} = \mathrm{e}^{\mathrm{i}Kt/2}\,\hat{A}\hat{A}^{+} + \hat{B}^{+}\left(\cos(\hat{f}_{\hat{n}}t) + \mathrm{i}K\frac{1}{\hat{f}_{\hat{n}}}\sin(\hat{f}_{\hat{n}}t)\right)\hat{B} \tag{49}$$

$$\hat{A} = \mathrm{i}g\hat{a}\frac{1}{\sqrt{g^2(\hat{a}^{+}\hat{a} + \hat{a}\hat{a}^{+})}} \tag{50}$$

$$\hat{B} = \mathrm{i}g\frac{1}{\sqrt{g^2(\hat{a}^{+}\hat{a} + \hat{a}\hat{a}^{+})}}\hat{a} \tag{51}$$

$$\hat{f}_{\hat{n}} = \sqrt{K^2/4 + g^2(\hat{a}^{+}\hat{a} + \hat{a}\hat{a}^{+})}. \tag{52}$$

The time evolution of the state vector is given by

$$|\psi(t)\rangle = \hat{T}^{+}\hat{U}\hat{T}|\psi(0)\rangle \tag{53}$$

where $|\psi(0)\rangle$ is the initial wave vector and $\hat{T} = \hat{T}_2\hat{T}_1$ has the form of

$$\hat{T} = \frac{1}{2}\begin{pmatrix} \hat{D}^{+}(2\alpha) & \sqrt{2} & \hat{D}(2\alpha) \\ \sqrt{2}\hat{D}^{+}(2\alpha) & 0 & -\sqrt{2}\hat{D}(2\alpha) \\ \hat{D}^{+}(2\alpha) & -\sqrt{2} & \hat{D}(2\alpha) \end{pmatrix}. \tag{54}$$

For an initial state $|\psi(0)\rangle = |1\rangle|\beta\rangle$, i.e. the inner state of the ions in the excited state and the external state in the coherent state, we compute the time evolution of the mean number of vibrational quanta which is given by

$$\langle\hat{n}\rangle = \langle\psi(\tau)|\hat{a}^{+}\hat{a}|\psi(\tau)\rangle \tag{55}$$

where $\tau = gt$ is the scaled time. The evolution of $\langle\hat{n}\rangle$ with the scaled time is shown in figure 1. We can see that the phenomenon of super-revivals exists in the two-ion system, as an analogy to the single-ion system [8] and the driven Jayness–Cummings model [14]. We note that this peculiar behaviour is sensitive to the phase of the initial coherent state $|\beta\rangle$. For instance, if we select $\beta = 5.0 + \mathrm{i}0.5$ and $\eta = 0.5$, we only observe ordinary revivals with the first revival time τ_R given by $\tau_R = 2\pi|\beta|$ (see figure 1(*a*)). However, for $\beta = 5.0 + \mathrm{i}0.5$ and $\eta = 0.5$, the super-revivals do occur for a scaled time $\tau_{SR} \approx 10\tau_R$ (see figure 1(*b*)).We would like to point out that the scaled period of super-revivals in the two-ion system will be small compared to the single-ion system (see [8]).

For the three-ion system, the Hamiltonian (37) may be represented as

$$\begin{aligned} H_I'' = {} & \left(\frac{3\Omega}{2} + \frac{3}{4}|\alpha|^2\nu\right)|1\rangle\langle 1| + \left(\frac{\Omega}{2} + \frac{7}{4}|\alpha|^2\nu\right)|2\rangle\langle 2| \\ & + \left(-\frac{\Omega}{2} + \frac{7}{4}|\alpha|^2\nu\right)|3\rangle\langle 3| + \left(-\frac{3\Omega}{2} + \frac{3}{4}|\alpha|^2\nu\right)|4\rangle\langle 4| \\ & + \left\{-\frac{\sqrt{3}}{2}\alpha\hat{a}|1\rangle\langle 2| - \alpha\hat{a}|2\rangle\langle 3| - \frac{\sqrt{3}}{2}\alpha\hat{a}|3\rangle\langle 4| + \mathrm{h.c.}\right\}. \end{aligned} \tag{56}$$

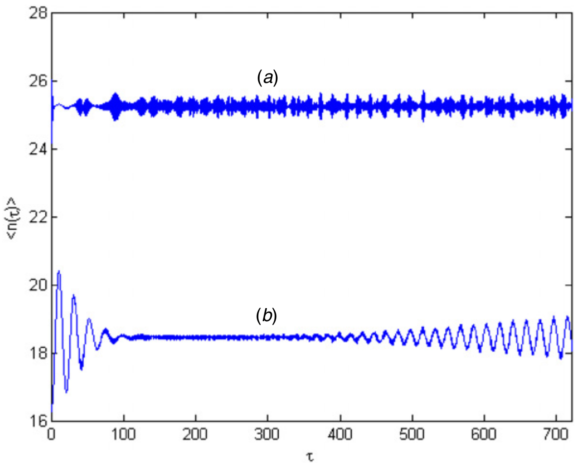

Figure 1. The evolution of the mean number of vibrational quanta as a function of the scaled time $\tau = gt$. (*a*) Ordinary revivals occurring for $\beta = 5.0 + i0.5$ and $\eta = 0.5$, and (*b*) super-revivals occurring for $\beta = 0.5 + \mathrm{i}5.0$ and $\eta = 0.5$.

This is a four-level cascade model in the text of cavity QED. We can obtain analytical solution of the evolution operator to this problem, but the expression of the evolution operator is rather complicated. For more than three ions system, the Hamiltonians (37) may be rewritten as

$$\tilde{\hat{H}}'' = \nu\hat{a}^{+}\hat{a} + \frac{\Omega}{2}\hat{J}_z + \frac{\eta^2\nu}{4}(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+) + \mathrm{i}\frac{\eta\nu}{2}(\hat{a}\hat{J}_+ - \hat{a}^{+}\hat{J}_-). \tag{57}$$

It can be seen that $\hat{H}_I''$ is a Dicke-type interaction Hamiltonian whose evolution operator is difficult to be derived analytically.

In conclusion, we have found that the system which consists of N two-level trapped ion interacting with a laser beam can be transformed into a multi-level Jaynes–Commings model by applying unitary transformations successively. As an example, we have investigated dynamics of the two-ion system, and found that the super-revivals will take place in that system for appropriate values of the parameters.

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